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ANALYSIS OF COMPRESSIBLE POTENTIAL FLOW OVER NONLIFTING AIRFOILS USING THE DUAL RECIPROCITY METHOD

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***Abstract**. The use of the linearized potential model for the analysis of compressible flows is quite popular, and provides good results for subsonic and supersonic flows. However, the calculation of airfoils and wings subject to transonic flows requires a non-linear model, such as the transonic small-disturbance (TSD) potential equation. The solution of the problem by a singularity distribution requires singularities over the field, as well as panels on the boundary, characterizing the procedure known as field panel method. The present work shows results of calculations of the transonic small-disturbance potential equation, with the use of the dual reciprocity method (DRM), which permits calculation of integrals only at the boundary of the problem, without the need of field distributions. This approach, compared to the field panel methods, takes considerably less computer time. The results show very good agreement with other methods found in literature.*

***Keywords.** transonic aerodynamics, panel methods, dual reciprocity method.*

1. Introduction

Calculation of subsonic and of supersonic flows over airfoils and wings by potential methods is possible with a linear theory, which provides accurate results for the pressure distribution. However, transonic flows cannot be modeled by a linear equation, and the linear potential theory, well developed for purely subsonic and supersonic flows, is not capable to describe a transonic flow.

Since Mach numbers of the transonic range are usually commercially interesting for transport aircraft, considerable attention has been given to the study of transonic flows. The use of an approach similar to the panel methods developed by Hess and Smith (1967) would be of much importance, since it allows the determination of pressure distribution over bodies of arbitrary geometry, without a need of extensive calculations. Panel methods have proven to be useful tools to design airfoils and wings, with less computer time required than Euler or Navier-Stokes calculations, since only calculations along the surface of the wings (the boundary of the problem) are required in a panel method formulation.

The work of Spreiter and Alksne (1955) is an attempt to solve the non-linear problem which is characteristic of transonic flows. The approach is based on the integral equation, and is similar to the panel methods mentioned. However, the application of Green's theorem to the transonic equation results in an integral equation, with integrals evaluated along the boundary and along the field. The solution of this equation is found by an approximation of the velocity profile along the field, which can reduce the integral along the field to an integral along the boundary. A similar method was developed by Nixon (1974), but with a different approximation of the velocity profile.

With faster computers available, it became possible to solve the integral equation with an integral along the domain, which resulted on field panel methods. Ribeiro and Soviero (1987) presented a field panel method for the solution of the transonic flow over an airfoil, based on the transonic small-disturbances equation. The results are very good; however, much more computer time is required by a field panel method than by a regular panel method.

A transformation of the domain integral to surface integrals is desired if one wishes to decrease the computer time required for the calculation of transonic flows. The appearance of the Dual Reciprocity Method (DRM), presented by Partridge et al (1992), gives a new alternative for the study of transonic flows. This method allows the calculation with panels distributed only along the boundary of the problem. The application of DRM for transonic flows was presented in the work of Uhl et al (1999), where wings at transonic speed had their pressure distributions calculated with good results.

This work focuses on the application of the Dual Reciprocity Method to airfoils, modeled by the transonic small-disturbances equation. This model is of interest because of its simplicity, since all the singularities are distributed along the chord. Rapid calculations can thus be performed, providing good estimations of the pressures on the airfoil surface.

2. Mathematical model

The full potential equation for a two-dimensional compressible flow can be written as:

$$\left(1-M_\infty{}^2\right)\phi_{xx}+\phi_{yy}=M_\infty{}^2\left[(\gamma+1)\frac{\phi_x}{V_\infty}+\frac{\gamma+1}{2}\frac{\phi_x{}^2}{V_\infty{}^2}+\frac{\gamma-1}{2}\frac{\phi_y{}^2}{V_\infty{}^2}\right]\phi_{xx}+$$
$$+M_\infty{}^2\left[(\gamma-1)\frac{\phi_x}{V_\infty}+\frac{\gamma-1}{2}\frac{\phi_x{}^2}{V_\infty{}^2}+\frac{\gamma+1}{2}\frac{\phi_y{}^2}{V_\infty{}^2}\right]\phi_{yy}+ \quad (1)$$
$$+M_\infty{}^2\left[\frac{\phi_y}{V_\infty}\left(1+\frac{\phi_x}{V_\infty}\right)\left(\phi_{xy}+\phi_{yx}\right)\right]$$

With the hypothesis of small disturbances, the right hand side of Eq. (1) can be approximated by zero if the undisturbed Mach number is less than 5, and is outside of the transonic range. In these cases, Eq. (1) becomes:

$$\left(1-M_\infty{}^2\right)\phi_{xx}+\phi_{yy}=0 \quad (2)$$

This is a linear partial differential equation, much simpler than Eq. (1). Its type depends on the undisturbed Mach number. If M_∞ is less than 1, which means the incident flow is subsonic, Eq. (2) is of the elliptic kind; if M_∞ is greater than 1 (supersonic incident flow), the equation is hyperbolic.

Equation (2) is not a good mathematical model for transonic flows over an airfoil. When a flow has Mach number close to unity, there are subsonic and supersonic zones in the flow. In order to describe the different character of the subsonic and supersonic zones, a mixed-type differential equation must be used. The simplest one is known as Transonic Small Disturbance (TSD) equation, and is presented in Eq. (3).

$$\left(1-M_\infty{}^2\right)\phi_{xx}+\phi_{yy}=\frac{(\gamma+1)M_\infty}{V_\infty}\phi_x\phi_{xx} \quad (3)$$

It is possible to define:

$$\beta^2=1-M_\infty{}^2 \quad (4)$$

$$k=\frac{(\gamma+1)M_\infty}{V_\infty} \quad (5)$$

Thus, Eq. (3) becomes:

$$\beta^2\phi_{xx}+\phi_{yy}=k\phi_x\phi_{xx} \quad (6)$$

The usual boundary conditions for eq. (6) are that values of the potential vanish at infinity, and that the velocity normal to the surface of the airfoil is zero. This second condition is linearized when small disturbances are assumed, and is satisfied at the chord. Written using the velocity potential, the boundary conditions at the surface are given in Eq. (7).

$$\frac{\partial y_S}{\partial x_S}=\frac{\phi_y}{V_\infty} \quad (7)$$

Since the coefficients of Eq. (3) are constant, it is possible to transform its variables, to obtain a simpler equation in a transformed plan. The transformed variables *X* and *Y* and the transformed potential Φ can be obtained using Eqs. (8)-(10).

$$X=x \quad (8)$$

$$Y=\beta y \quad (9)$$

$$\Phi=\frac{k}{\beta^2}\phi \quad (10)$$

Equation (6) in the transformed plan becomes:

$$\Phi_{XX} + \Phi_{YY} = \nabla^2 \Phi = \Phi_X \Phi_{XX} \tag{11}$$

The equation above is in the transformed plan. The boundary conditions also need to be transformed; the resulting boundary conditions at the transformed plan are given in Eq. (12).

$$\frac{\partial y_S}{\partial x_S} = \frac{1}{V_\infty} \frac{\beta^3}{k} \Phi_Y \tag{12}$$

Equation (11) is non-linear, and its solution is more difficult than that of the linear problem. The application of Green's theorem with eq. (11) leads to eq. (13).

$$c\Phi(x_i) = -\int \frac{\partial \Phi}{\partial n} G d\Gamma + \int \Phi \frac{\partial G}{\partial n} d\Gamma + \iint \Phi_X \Phi_{XX} G d\Omega \tag{13}$$

In the equation above, G is the fundamental solution of Laplace's equation, which is, for the two-dimensional case, $\ln r$, being r the distance between x_i, the point of interest, and any other point x. The first two integrals are calculated at the boundary of the problem. However, the third integral is calculated at the field.

The solution of the problem described by Eq. (11) can be found using a field panel method; this solution was found by Ribeiro and Soviero (1987). This approach is similar to the panel method proposed by Hess and Smith (1967), but the field panel method requires a distribution of sources in the field and of sources and dipoles on the surface of the airfoil. These distributions are done by a proper discretization of the domain on panels; consequently, a field panel method for a two-dimensional problem consists on a division of the surface of the airfoil into segments with unknown distributions of sources and dipoles, and a division of the field into small areas with unknown source distributions. The distributions can be found by an iterative process.

The requirement of panels in the field to calculate the potential is due to the presence of the double integral in Eq. (13). Since panels must be distributed in the field, there is a notable increase on computer time for the calculations. Panel methods for incompressible flow require only panels at the surface of the airfoil. Since we have to deal with a much greater number of unknown distributions of singularities, a solution of a transonic flow with a field panel method takes much more time than an incompressible solution found with a panel method based on Hess and Smith's work.

The Dual Reciprocity Method (DRM), such as presented by Partridge et al (1992), can be applied to the present problem, in order to eliminate the double integral in Eq. (13). With only integrals calculated at the surface of the airfoil, it is not required to divide the field into panels. Consequently, the solution takes less computer time.

If the right hand side of Eq. (11) can be expressed by

$$\Phi_X \Phi_{XX} = \sum_{j=1}^{n} \alpha_j f_j \ , \tag{14}$$

being f_j a set of n appropriate functions, and being ϕ_j functions defined as

$$\nabla^2 \phi_j = f_j \ , \tag{15}$$

it is possible to apply Green's theorem with ϕ_j, resulting on Eq. (16).

$$c\phi_j(x_i) = -\int \frac{\partial \phi_j}{\partial n} G d\Gamma + \int \phi_j \frac{\partial G}{\partial n} d\Gamma + \iint f_j G d\Omega \tag{16}$$

Equation (16) is valid for the whole set of functions defined on Eqs. (14) and (15). Consequently, it is possible to write Eq. (17).

$$\sum_{j=1}^{n} \alpha_j c\phi_j(x_i) = \sum_{j=1}^{n} \alpha_j \left(-\int \frac{\partial \phi_j}{\partial n} G d\Gamma + \int \phi_j \frac{\partial G}{\partial n} d\Gamma \right) + \iint \sum_{j=1}^{n} \alpha_j f_j G d\Omega \tag{17}$$

Using the definition of the functions f_j, Eq. (17) becomes

$$\sum_{j=1}^{n}\alpha_j c\phi_j(x_i)=\sum_{j=1}^{n}\alpha_j\left(-\int\frac{\partial\phi_j}{\partial n}Gd\Gamma+\int\phi_j\frac{\partial G}{\partial n}d\Gamma\right)+\iint\Phi_X\Phi_{XX}Gd\Omega\ . \tag{18}$$

Thus, the double integral of Eq. (13) can be transformed to line integrals.

$$\iint\Phi_X\Phi_{XX}Gd\Omega=\sum_{j=1}^{n}\alpha_j c\phi_j(x_i)+\sum_{j=1}^{n}\alpha_j\left(\int\frac{\partial\phi_j}{\partial n}Gd\Gamma-\int\phi_j\frac{\partial G}{\partial n}d\Gamma\right) \tag{19}$$

With equations (13) and (19), it is possible to write de potential at the point xi with only integrals calculated at the boundary of the problem, as shown in Eq. (20).

$$c\Phi(x_i)=-\int\frac{\partial\Phi}{\partial n}Gd\Gamma+\int\Phi\frac{\partial G}{\partial n}d\Gamma+\sum_{j=1}^{n}\alpha_j c\phi_j(x_i)+\sum_{j=1}^{n}\alpha_j\left(\int\frac{\partial\phi_j}{\partial n}Gd\Gamma-\int\phi_j\frac{\partial G}{\partial n}d\Gamma\right) \tag{20}$$

Equation (20) allows the determination of the velocity potential of transonic flows. However, new unknowns (α_j) appear at the integral equation. Once the values of the interpolation coefficients α_j are determined, the potential can be calculated in the whole domain.

In this work, only symmetrical nonlifting airfoils were analyzed. Thus, Eq. (20) can be solved without distributions of normal dipoles; this simplifies the problem, leading to Eq. (21), which is the final form of the integral equation to be solved.

$$c\Phi(x_i)=-\int\frac{\partial\Phi}{\partial n}Gd\Gamma+\sum_{j=1}^{n}\alpha_j\left[c\phi_j(x_i)+\int\frac{\partial\phi_j}{\partial n}Gd\Gamma\right] \tag{21}$$

The expression presented above shows that the velocity potential of a nonlifting transonic flow can be calculated with two additional terms. These terms come from the dual reciprocity method. It should be noted that the first term of Eq. (21) is identical to that of the linear theory. Consequently, the distribution of sources along the chord can be calculated with linear theory. The calculation of the remaining two terms is explained in the next section.

3. Numerical implementation

The functions f_j employed in the Dual Reciprocity Method can be any set of functions that can provide an adequate expansion of the right hand side of Eq. (11). Uhl et al (1999) have proposed the functions

$$f_j=1+r \tag{22}$$

as the interpolation functions.

Application of Eq. (15) with the chosen functions results

$$\phi_j=\frac{r^2}{4}+\frac{r^3}{9} \tag{23}$$

The n functions defined in Eqs. (22) and (23) are centered at n nodes distributed at the boundary and at the field. These nodes are chosen in order to represent well the variations of $\Phi_X\Phi_{XX}$ along the domain with the set of f_j functions. The functions chosen are very simple, but can provide good expansions of the non-linear part of Eq. (11).

If the values of $\Phi_X\Phi_{XX}$ are calculated at the n nodes, it is possible to write a system with n equations and n unknowns.

$$\begin{bmatrix}\Phi_X\Phi_{XX}(x_1)\\ \Phi_X\Phi_{XX}(x_2)\\ \vdots\\ \Phi_X\Phi_{XX}(x_n)\end{bmatrix}=\begin{bmatrix}f_1(x_1) & f_2(x_1) & \cdots & f_n(x_1)\\ f_1(x_2) & f_2(x_2) & \cdots & f_n(x_2)\\ \vdots & \vdots & \ddots & \vdots\\ f_1(x_n) & f_2(x_n) & \cdots & f_n(x_n)\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{bmatrix} \tag{24}$$

If the f matrix is inverted, it can be used to determine the values of the interpolation coefficients αj based on the calculated values of $\Phi_X\Phi_{XX}$.

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_n(x_1) \\ f_1(x_2) & f_2(x_2) & \cdots & f_n(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ f_1(x_n) & f_2(x_n) & \cdots & f_n(x_n) \end{bmatrix}^{-1} \begin{bmatrix} \Phi_X \Phi_{XX}(x_1) \\ \Phi_X \Phi_{XX}(x_2) \\ \vdots \\ \Phi_X \Phi_{XX}(x_n) \end{bmatrix} \tag{25}$$

Based on this approach, it is possible to define the calculation steps.

a. The intensities of the sources distributed along the chord are calculated according to linear theory. The solution is considered a first guess for the potential in the whole field;
b. The right hand side of eq. (10) is calculated based on the current calculated potential;
c. Values of α_j are calculated with eq. (25).
d. A new potential can be obtained if the values of α_j are used with eq. (x);
e. This new potential becomes the current one, and steps b-d are repeated until convergence is attained.

The method outlined above has a rapid convergence for Mach numbers below the critical value. When M_∞ is increased above its critical value, and supersonic regions appear in the flow, the iterative process diverges. In order to allow convergence for higher Mach numbers, finite-difference methods make use of schemes with different forms of calculation for subsonic and supersonic velocities. Ribeiro and Soviero (1987) suggest the use of a function of artificial viscosity for the calculation of finite differences at points with supersonic velocities, in order to allow the convergence of the method at higher Mach numbers and to provide a good representation of zones with large velocity gradients, which represent shock waves. It should be noted that this method avoids the appearance of expansion shocks. The finite-difference scheme becomes, for points with local Mach number greater than one:

$$\sigma(x_i) = \Phi_X \Phi_{XX}(x_i) - \nu(x_i)[fv(x_i)\Phi_{XX}(x_i)] + \nu(x_{i-1})[fv(x_{i-1})\Phi_{XX}(x_{i-1})] \tag{26}$$

In Equation (26), x_{i-1} is a point upwind of the reference point x_i; consequently, the scheme presented in Eq. (26) adds a term of upwind differences in order to represent adequately a supersonic flow. The function *fv* is an arbitrary function of artificial viscosity. In the present work, *fv* was given by (Φx-1), which was the expression used by Ribeiro and Soviero (1987); the same function was chosen to provide a validation of the present method with the results shown on that work. The function *v* is equal to 1 for points with supersonic speeds, and equal to 0 for points with subsonic speeds.

Equation (26) is an example of a conservative scheme of artificial viscosity. It is possible to use a non-conservative scheme, such as the one given by Eq. (27).

$$\sigma(x_i) = \Phi_X \Phi_{XX}(x_i) - \nu(x_i)[fv(x_i)\Phi_{XX}(x_i) - fv(x_{i-1})\Phi_{XX}(x_{i-1})] \tag{27}$$

5. Results

The calculations were performed using a circular biconvex airfoil, with 6 percent of maximum thickness. Figure (1) shows the airfoil, as well as the nodes distributed along the domain.

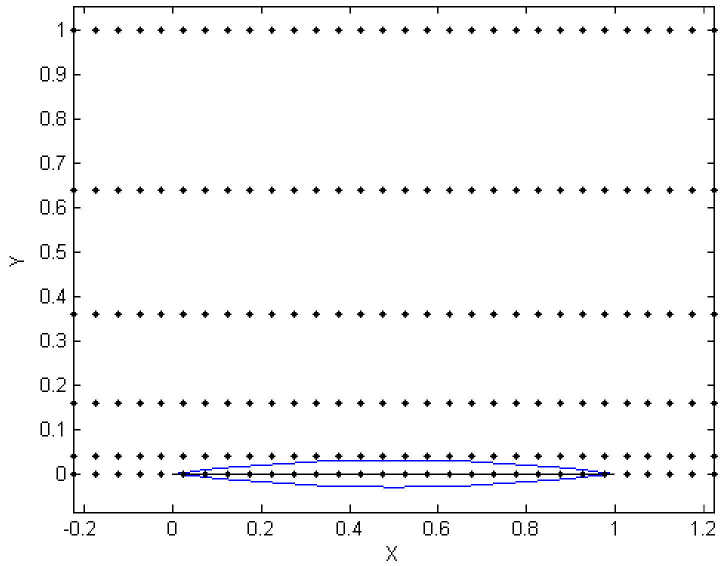

Figure 1. Discretization of the airfoil and position of the nodes.

The nodes shown on Fig. (1) are equally spaced along the X axis, and are distributed using a quadratic function along the Y axis, allowing a node clustering near the surface of the airfoil.

In the present work, the chord of the airfoil was divided into 20 panels, with control points placed at the center of each panel. The intensities of the sources are calculated in order to fulfill the boundary conditions at the control points. It should be noted that there are nodes that are coincident with the control points. This coincidence does not cause any problem with the numerical calculations, and provide a better representation of the nonlinear term of the governing equation.

The nodes were distributed only along the positive *y* direction, taking advantage of the symmetry of the problem in order to decrease computer time. The results of the present work were obtained using six nodes along the *y* axis, extended until one chord above the *x* axis.

In order to validate the method, a comparison with the field panel method of Ribeiro and Soviero (1987) was made. Since the work performed by this reference was based on the same equation, and the same airfoil was used, its results can be used to test if the dual reciprocity method is valid.

The pressure coefficient can be obtained with the expression for small disturbances, presented on Eq. (28).

$$C_P = -\frac{2\phi_x}{V_\infty} \tag{28}$$

The non-linear expression of the pressure coefficient was also tested, with little change on the values of C_P. This was expected, for the disturbances are small; consequently, the application of Eq. (28) is recommended for its simplicity.

Figures (2) to (4) show the comparison between dual reciprocity and field panel methods, for Mach numbers equal to 0.8, 0.85 and 0.87.

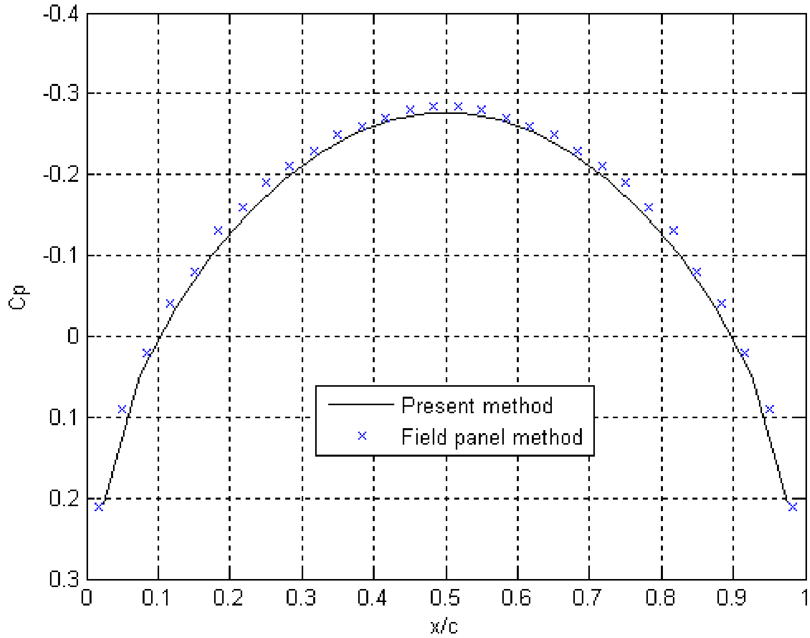

Figure 2. Pressure distribution for the circular-arc profile, M=0.8.

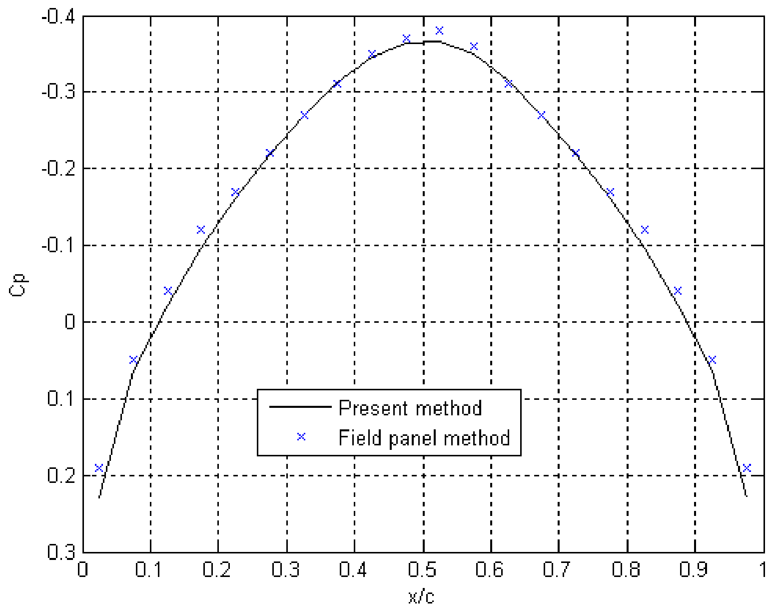

Figure 3. Pressure distribution for the circular-arc profile, M=0.85.

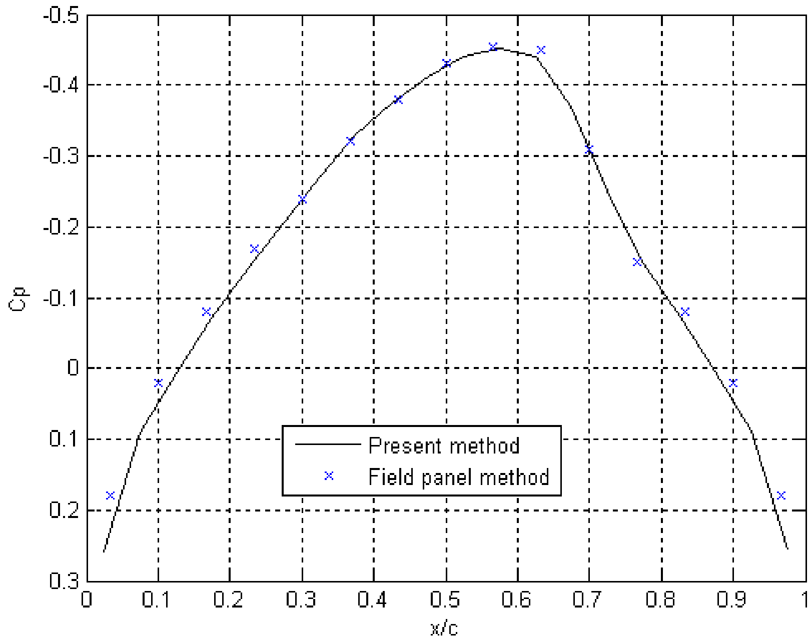

Figure 4. Pressure distribution for the circular-arc profile, M=0.87.

The results show good agreement between both methods. The differences are probably due to the nature of each method. The field panel method takes the source intensity at a control point and uses it as a constant value for the whole panel; and the dual reciprocity method uses interpolation functions, and thus gives a different approximation to the distribution along the field. However, it should be noted that the results are very similar, showing that the dual reciprocity method can be a substitute for field panel methods, giving accurate results that require less computer time.

The results shown on Figs. (3) and (4) are characteristic of transonic flows, since the minimum value of CP is more negative than its critical value. Convergence for both methods was obtained using the conservative version given by Eq. (26).

Figure (5) shows the results of the method for Mach number equal to 0.806, as well as experimental results taken from the work of Knechtel (1959).

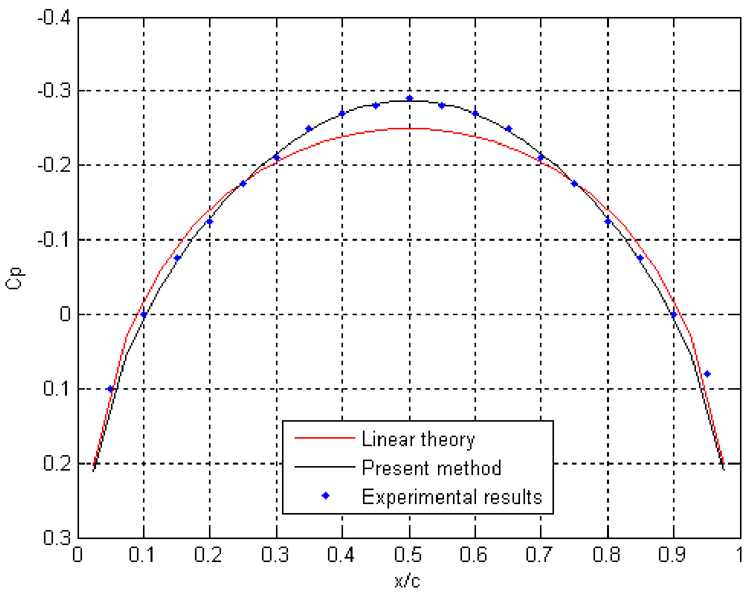

Figure 5. Pressure distribution on circular biconvex profile at Mach number 0.806.

The flow surrounding the airfoil in this case is entirely subsonic; thus, for the results presented on Fig. (5), the scheme presented by Eq. (26) was not used. Examination of Fig. (2) shows that the results of the present method agree very closely with the experimental values obtained by Knechtel (1959).

The close agreement found here cannot be expected to happen every time, since real flows have viscous effects, and their magnitude can cause differences between pressure distributions found by experiment and by the present method. However, the results shown on Fig, (5) show that the dual reciprocity method captures very well the tendencies of the real flow. The present method gives a more accurate value of the minimal C_P, which is valuable for the determination of the critical Mach number; and provides a better representation of the shape of the pressure distribution.

Figure (6) shows the results for Mach 0.861 for the conservative version, based on Eq. (26); and Figure (7) shows the results for the same Mach number, but with the non-conservative version, with calculations done such as shown on Eq. (27).

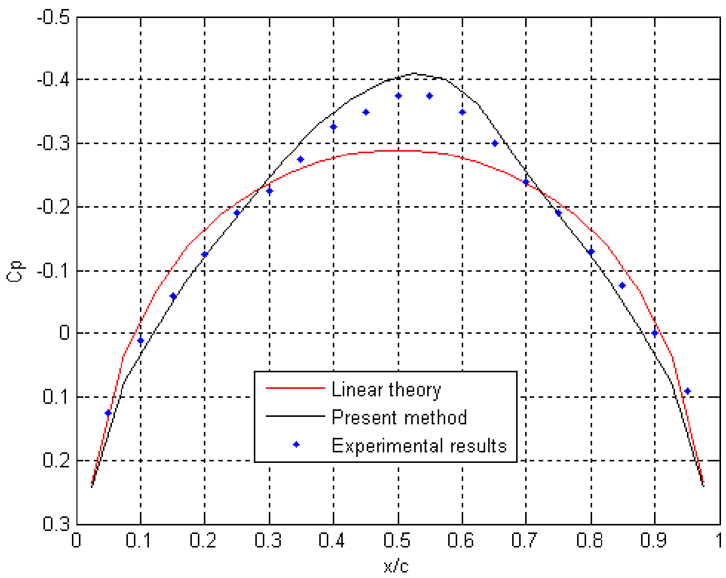

Figure 6. Pressure distribution on circular biconvex profile at Mach number 0.861 – conservative version.

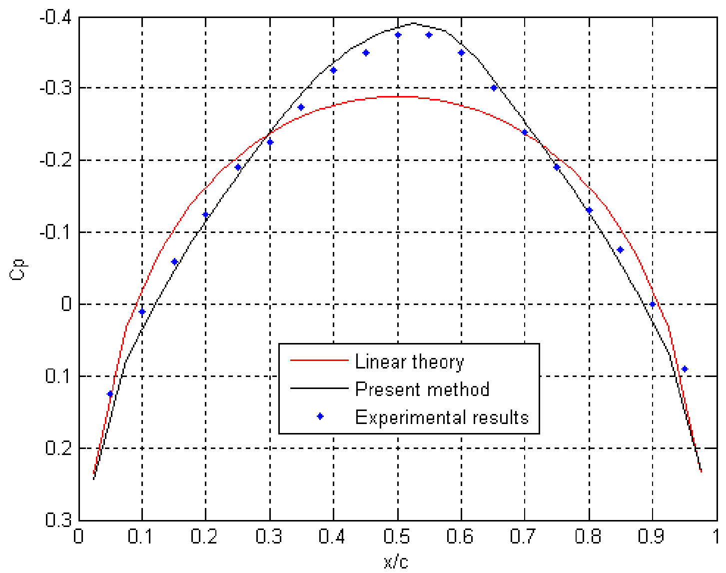

Figure 7. Pressure distribution on circular biconvex profile at Mach number 0.861 – non-conservative version.

The results show that the present method provides a good prediction of the pressure distribution over the airfoil. Linear theory prove to be inadequate for transonic flows, giving a shape of the distribution which is not close to the actual one.

The non-conservative version gives results closer to the experimental values. However, the conservative version gives a steeper pressure gradient on the region of the shock wave. The comparison of the two versions for a Mach number of 0.87 is shown on Fig. (8).

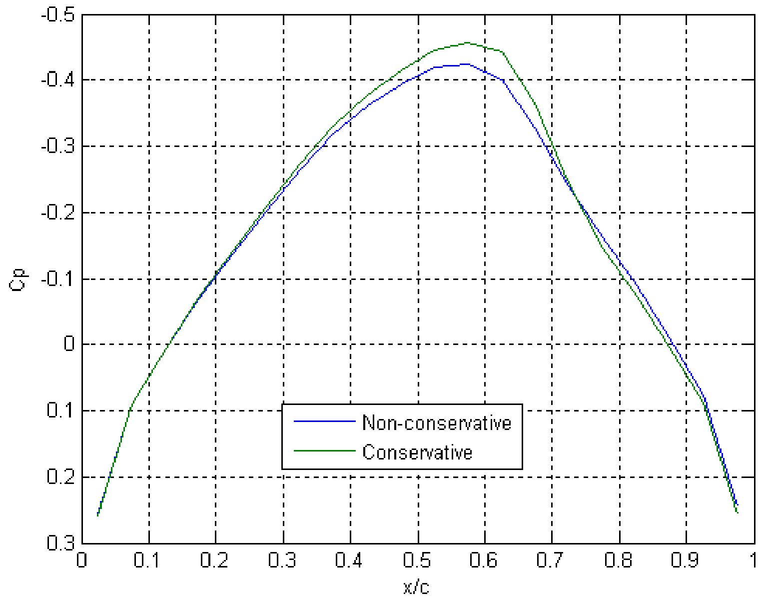

Figure 8. Comparison between non-conservative and conservative versions.

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